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MICROECONOMIC THEORY

BASIC PRINCIPLES AND EXTENSIONS

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The n -input case

Generalizing the elasticity of substitution to the many-input case raises several complications. One approach is to adopt a definition analogous to Equation 7.33; that is, to define the elasticity of substitution between two inputs to be the proportionate change in the ratio of the two inputs to the proportionate change in the *RTS* between them while holding output constant.⁷ To make this definition complete, it is necessary to require that all inputs other than the two being examined be held constant. However, this latter requirement (which is not relevant when there are only two inputs) restricts the value of this potential definition. In real-world production processes, it is likely that any change in the ratio of two inputs will also be accompanied by changes in the levels of other inputs. Some of these other inputs may be complementary with the ones being changed, whereas others may be substitutes, and to hold them constant creates a rather artificial restriction. For this reason, an alternative definition of the elasticity of substitution that permits such complementarity and substitutability in the firm's cost function is generally used in the n -good case. We will describe this alternative concept in the next chapter.

Four simple production functions

In this section we illustrate four simple production functions, each characterized by a different elasticity of substitution. These are shown only for the case of two inputs, but generalization to many inputs is easily accomplished (see the Extensions for this chapter).

Case 1: Linear ($\sigma = \infty$)

Suppose that the production function is given by

$$q = f(k, l) = \alpha k + \beta l. \quad (7.34)$$

It is easy to show that this production function exhibits constant returns to scale: For any $\tau > 1$,

$$f(\tau k, \tau l) = \alpha \tau k + \beta \tau l = \tau(\alpha k + \beta l) = \tau f(k, l). \quad (7.35)$$

All isoquants for this production function are parallel straight lines with slope $-\beta/\alpha$. Such an isoquant map is pictured in panel (a) of Figure 7.4. Because, along any straight-line isoquant, the *RTS* is constant, the denominator in the definition of σ (Equation 7.33) is equal to 0, and hence σ is infinite. Although this linear production function is a useful example, it is rarely encountered in practice because few production processes are characterized by such a case of substitution. Indeed, in this case capital and labor can be thought of as perfect substitutes for each other. An industry characterized by such a production function could use *only* capital or *only* labor, depending on these inputs' prices. It is hard to envision such a production process: Every machine needs someone to press its buttons, and every laborer requires some capital equipment, however modest.

Case 2: Fixed proportions ($\sigma = 0$)

The production function characterized by $\sigma = 0$ is the important case of a *fixed-proportions production function*. Capital and labor must always be used in a fixed ratio. The isoquants for this production function are L-shaped and are pictured in panel (b) of

⁷That is, the elasticity of substitution between input i and input j might be defined as

$$\sigma_{ij} = \frac{\partial \ln\left(\frac{x_i}{x_j}\right)}{\partial \ln\left(\frac{f_i}{f_j}\right)}$$

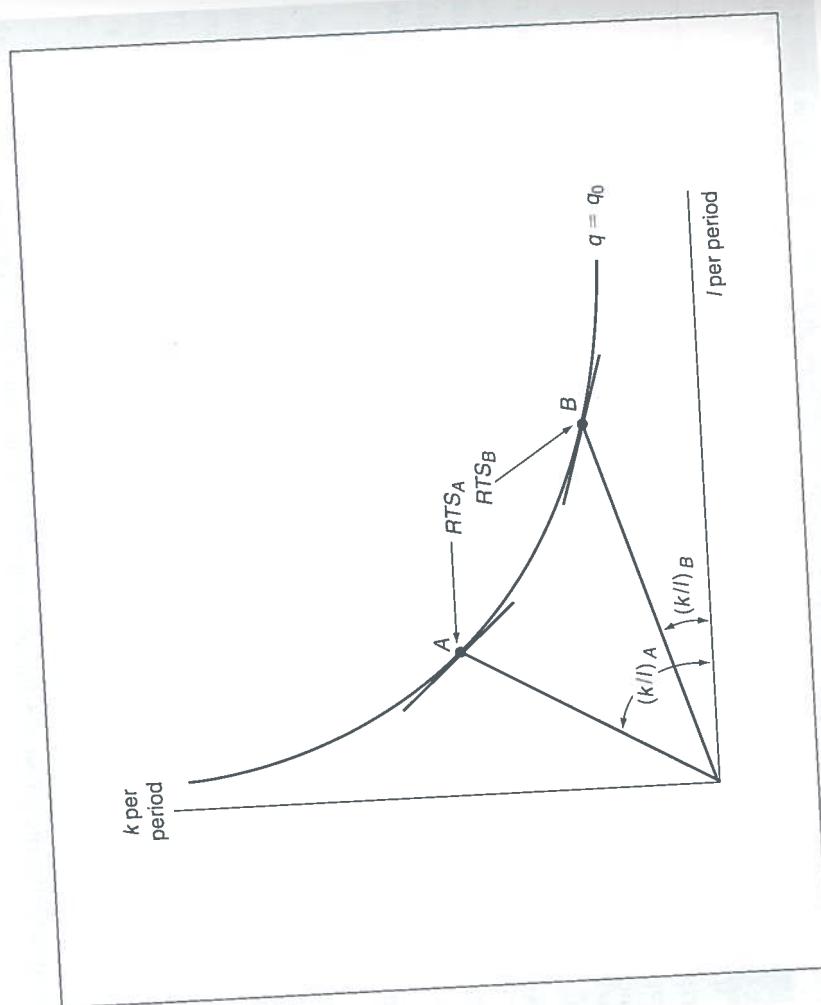
^aThe elasticity of substitution can be phrased directly in terms of the production function and its derivatives in the constant σ .
 turns to scale case as

$$\sigma = \frac{f_k \cdot f_l}{f \cdot f_{k,l}}$$

^aThe elasticity of substitution is quite cumbersome. Hence usually the logarithmic definition in Equation 7.33 is easiest to apply. For a compact summary, see P. Berck and K. Sydsæter, *Economist's Mathematical Manual* (Berlin: Springer-Verlag, 1999), Chapter 5.

FIGURE 7.3 Graphic Description of the Elasticity of Substitution

In moving from point A to point B on the $q = q_0$ isoquant, both the capital-labor ratio (k/l) and the *RTS* will change. The elasticity of substitution (σ) is defined to be the ratio of these proportional changes. It is a measure of how curved the isoquant is.



raise output, and hence the marginal product of labor is zero; additional labor is superfluous in this case. Similarly, if $ak > bl$, labor is the binding constraint on output and additional capital is superfluous. When $ak = bl$, both inputs are fully utilized. When this happens, $k/l = b/a$, and production takes place at a vertex on the isoquant map. If both inputs are costly, this is the only cost-minimizing place to operate. The locus of all such vertices is a straight line through the origin with a slope given by b/a .

The fixed-proportions production function has a wide range of applications.⁸ Many machines, for example, require a certain number of people to run them, but any excess labor is superfluous. Consider combining capital (a lawn mower) and labor to mow a lawn. It will always take one person to run the mower, and either input without the other is not able to produce any output at all. It may be that many machines are of this type and require a fixed complement of workers per machine.⁹

Case 3: Cobb-Douglas ($\sigma = 1$)

The production function for which $\sigma = 1$, called a *Cobb-Douglas production function*¹⁰ provides a middle ground between the two polar cases previously discussed. Isoquants for the Cobb-Douglas case have the “normal” convex shape and are shown in panel (c) of Figure 7.4. The mathematical form of the Cobb-Douglas production function is given by

$$q = f(k, l) = ak^a l^b, \quad (7.37)$$

where A , a , and b are all positive constants.

The Cobb-Douglas function can exhibit any degree of returns to scale, depending on the values of a and b . Suppose all inputs were increased by a factor of t . Then

$$\begin{aligned} f(tk, tl) &= A(tk)^a (tl)^b = at^a t^b k^a l^b \\ &= t^{a+b} f(k, l). \end{aligned} \quad (7.38)$$

Hence, if $a + b = 1$, the Cobb-Douglas function exhibits constant returns to scale, because output also increases by a factor of t . If $a + b > 1$, the function exhibits increasing returns to scale, whereas $a + b < 1$ corresponds to the decreasing returns-to-scale case. It is a simple matter to show that the elasticity of substitution is 1 for the Cobb-Douglas function.¹¹ This fact has led researchers to use the constant-returns-to-scale version of the function for a general description of aggregate production relationships in many countries.

⁸With the form reflected by Equation 7.35, the fixed-proportions production function exhibits constant returns to scale, because

$$f(tk, tl) = \min(tk, tl) = t \cdot \min(ak, bl) = tf(k, l)$$

for any $t > 1$. As before, increasing or decreasing returns can be easily incorporated into the functions by using a nonlinear transformation of this functional form, such as $[f(k, l)]^\gamma$ where γ may be greater than or less than one.

⁹The lawn mower example points up another possibility, however. Presumably there is some leeway in choosing what size of lawn mower to buy. Hence, prior to the actual purchase, the capital-labor ratio in lawn mowing can be considered variable: Any device, from a pair of clippers to a gang mower, might be chosen. Once the mower is purchased, however, the capital-labor ratio becomes fixed.

¹⁰Named after C. W. Cobb and P. H. Douglas. See P. H. Douglas, *The Theory of Wages* (New York: Macmillan Co., 1934), pp. 132–35.

¹¹For the Cobb-Douglas,

$$RTS = \frac{f_l}{f_k} = \frac{ba k^a l^{b-1}}{a A k^{a-1} l^b} = \frac{b}{a} \frac{k}{l}$$

or

$$\ln RTS = \ln \left(\frac{b}{a} \right) + \ln \left(\frac{k}{l} \right).$$

Hence:

$$\sigma = \frac{\partial \ln k/l}{\partial \ln RTS} = 1.$$

FIGURE 7.4 Isoquant Maps for Simple Production Functions with Various Values for σ

Three possible values for the elasticity of substitution are illustrated in these figures. In (a), capital and labor are perfect substitutes. In this case the RTS will not change as the capital-labor ratio changes. In (b), the fixed-proportions case, no substitution is possible. The capital-labor ratio is fixed at b/a . A case of limited substitutability is illustrated in (c).

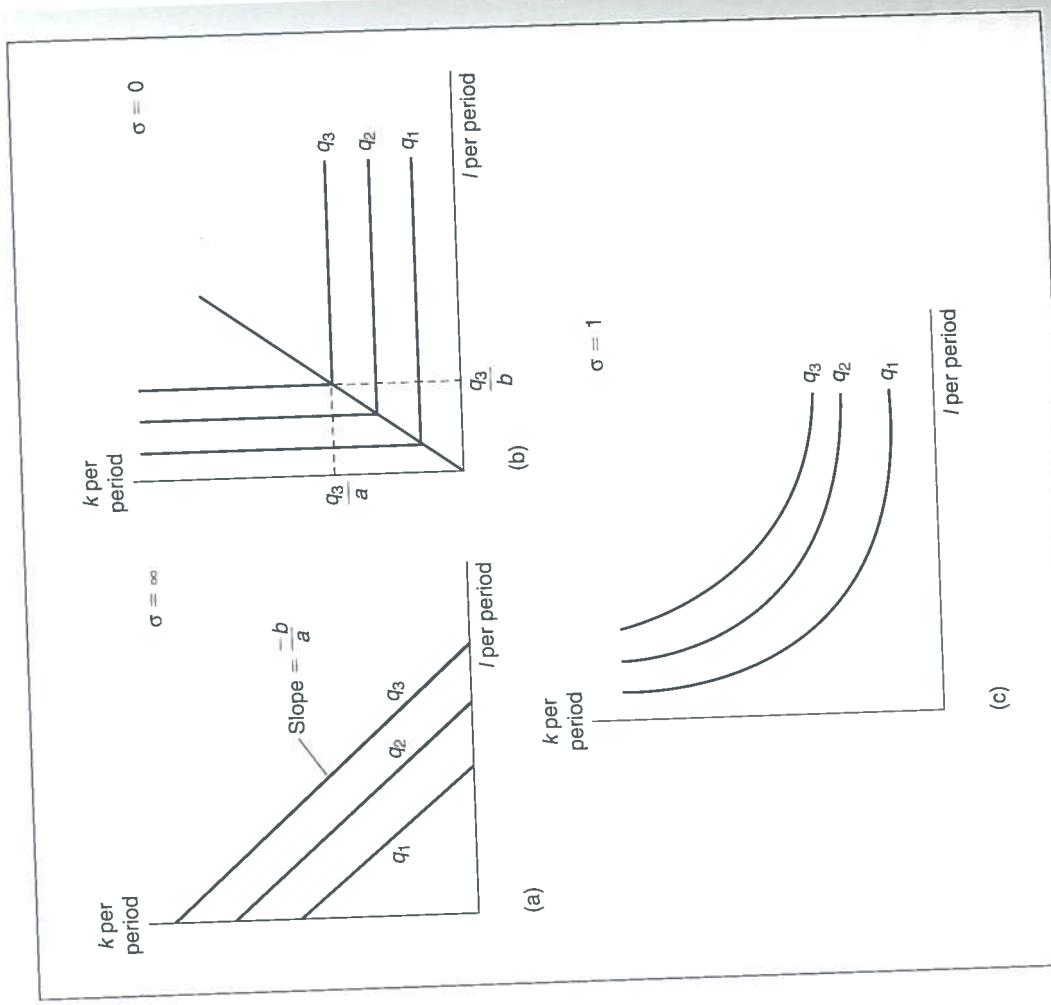


Figure 7.4. A firm characterized by this production function will always operate along the ray where the ratio k/l is constant. To operate at some point other than at the vertex of the isoquants would be inefficient, because the same output could be produced with fewer inputs by moving along the isoquant toward the vertex. Because k/l is a constant, it is easy to see from the definition of the elasticity of substitution that σ must equal 0.

The mathematical form of the fixed-proportions production function is given by

$$q = \min(ak, bl) \quad a, b > 0, \quad (7.36)$$

where the operator “min” means that q is given by the smaller of the two values in parentheses. For example, suppose that $ak < bl$; then $q = ak$, and we would say that capital is the binding constraint in this production process. The employment of more labor would not